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On double gauging of $U(1)$ symmetry on noncommutative space

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Abstract

We point out that a field φ charged under a global $U(1)$ symmetry generally allows for a starred localized extension with the transformation rule, $\varphi \to U_L \star \varphi \star U_R^{-1}$. This results in a double gauging of the global $U(1)$ symmetry on noncommutative space. We interpret the gauge theory so obtained in terms of the gauge fields that in the commutative limit appear naturally and are respectively the gauge field responsible for the charge and a decoupled vector field. The interactions are shown to be very different from those obtained by assigning a transformation rule of $\varphi \to U \star \varphi$ or $\varphi \star U^{-1}$.

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Charge is a global property of fields. Its conservation implies a global $U(1)$ symmetry. When the symmetry is gauged on ordinary spacetime, the charge of a field specifies how the field transforms locally together with the gauge field. This gauging procedure may be generalized to field theory on noncommutative (NC) spacetime in the approach of the Moyal-Weyl correspondence [1]. For a $U(1)$ gauge field transforming as

$$A_\mu(x) \quad\rightarrow\quad U(x)\star A_\mu(x)\star U^{-1}(x)-\frac{i}{e}U(x)\star\partial_\mu U^{-1}(x), \tag{1}$$

the following types of matter field transformation rules have been studied [2],

$$\begin{array}{rcl} \psi_+(x) & \rightarrow & U(x)\star\psi_+(x), \\ \psi_-(x) & \rightarrow & \psi_-(x)\star U^{-1}(x), \\ \psi_0(x) & \rightarrow & U(x)\star\psi_0(x)\star U^{-1}(x), \end{array} \tag{2}$$

besides the trivial identity representation. Here the Moyal star product is defined as

$$(f_1\star f_2)(x) \quad=\quad \left[\exp\left(\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu\right)f_1(x)f_2(y)\right]_{y=x}, \tag{3}$$

with $\theta_{\mu\nu}$ being the parameter characterizing the NC spacetime, and $U(x)$ is the starred exponential,

$$U(x)=\exp_\star[ie\alpha(x)]=1+ie\alpha(x)+\frac{(ie)^2}{2!}\alpha\star\alpha(x)+\cdots. \tag{4}$$

The restrictions on transformation rules and also on gauge groups originate essentially from the closure requirement of group multiplication and the noncommutativity of the star product between c-number functions [3, 4]. This poses an obstacle in realistic model building [5].

In this work we would like to point out a new transformation rule for matter fields that is more general than the ones listed in eqn. (2),

$$\varphi(x) \quad\rightarrow\quad U_L(x)\star\varphi(x)\star U_R^{-1}(x), \tag{5}$$

where $U_{L,R}(x)$ are two independent starred exponentials. This generalization is based on the following observation. The fact that the φ field carries a conserved, additive charge is strong enough to fix uniquely its local transformation rule on ordinary spacetime but not so on NC spacetime where the commutative point-wise multiplication is replaced by the noncommutative star product. The order of factors becomes relevant as we already saw in the transformation rules for the above ψ fields. On the other hand, as far as charge is concerned, the requirement that must be met is the global transformation rule of the

charged field for which there is no difference between the point-wise and star product. This is indeed the case for the rule in eqn. (5): for constant $U_{L,R}$, only the combination $U = U_L U_R^{-1}$ is relevant. Furthermore, multiplying more factors like $U_{L,R}(x)$ from the left or right in eqn. (5) is ambiguous because the order of these factors, while relevant due to the star, is not a well-defined concept [5]. This makes eqn. (5) the most general transformation that can be assumed for a charged field.

The implementation of the transformation rule (5) necessarily demands two $U(1)$ gauge bosons. The NC $U(1)\times U(1)$ gauge theory has been studied previously [6] by assigning to a scalar field two independent charges corresponding to the two factors of $U(1)$, which are then spontaneously broken. We stress that our point of view is quite different in this work. We assume for a field φ only one charge instead of two independent ones while introducing two sets of gauge bosons. This is not as surprising as it seems to be at first sight since it appears in some sense already for the neutral field ψ_0: we can introduce a gauge field for a matter field with no charge at all. This possibility is offered by the spacetime structure as opposed to field theory on ordinary spacetime in which an uncharged field cannot be engaged in gauge interactions. To clarify our point and show its consequences, we take the φ^4 theory as an example, but generalization to fermion fields is straightforward.

Consider the complex φ^4 theory on NC spacetime whose Lagrangian density is

$$\mathcal{L} = \partial_\mu\varphi\star\partial^\mu\varphi^\dagger - m^2\varphi\star\varphi^\dagger - \frac{\lambda}{2}\varphi\star\varphi^\dagger\star\varphi\star\varphi^\dagger. \tag{6}$$

Its equations of motion are derived by the variational principle generalized to NC spacetime (see Ref. [7] for example) as

$$\begin{aligned}\delta S &= \int d^4x\ \delta\mathcal{L}\\ &= \int d^4x\Big\{-\delta\varphi\star(\partial^2+m^2)\varphi^\dagger-(\partial^2+m^2)\varphi\star\delta\varphi^\dagger\\ &\quad -\frac{\lambda}{2}\Big(\delta\varphi\star\varphi^\dagger\star\varphi\star\varphi^\dagger+\varphi\star\varphi^\dagger\star\delta\varphi\star\varphi^\dagger\\ &\qquad +\varphi\star\delta\varphi^\dagger\star\varphi\star\varphi^\dagger+\varphi\star\varphi^\dagger\star\varphi\star\delta\varphi^\dagger\Big)\Big\}.\end{aligned} \tag{7}$$

Using the cyclicity property of integrals of star products, the interaction terms can be combined and one star can be ignored in each term,

$$\begin{aligned}\delta S &= -\int d^4x\Big\{\delta\varphi\left((\partial^2+m^2)\varphi^\dagger+\lambda\varphi^\dagger\star\varphi\star\varphi^\dagger\right)\\ &\qquad +\left((\partial^2+m^2)\varphi+\lambda\varphi\star\varphi^\dagger\star\varphi\right)\delta\varphi^\dagger\Big\}.\end{aligned} \tag{8}$$

The equations of motion are then,

$$(\partial^2+m^2)\varphi = -\lambda\varphi\star\varphi^\dagger\star\varphi \tag{9}$$

and its Hermitian conjugate.

The φ^4 theory in eqn. (6) has a global $U(1)$ symmetry. Now we construct its conserved currents by the Noether procedure. For this purpose, we make the global transformation star localized. As discussed above, this is not unique. Consider first an infinitesimal transformaion from the right with $U_R(x) = \exp_\star[-i\alpha_R(x)] = 1 - i\alpha_R(x) + \cdots$. Note that only the kinetic term contributes to the variation of the action. Using again the cyclicity property and ignoring one star in each term, we have

$$\delta S = \int d^4x\ \partial^\mu \alpha_R\ i(\partial_\mu \varphi^\dagger \star \varphi - \varphi^\dagger \star \partial_\mu \varphi) + O(\alpha_R^2), \tag{10}$$

which determines the right current to be

$$J_\mu^R = -i(\partial_\mu \varphi^\dagger \star \varphi - \varphi^\dagger \star \partial_\mu \varphi). \tag{11}$$

The left current is similarly obtained from the left transformation,

$$J_\mu^L = +i(\partial_\mu \varphi \star \varphi^\dagger - \varphi \star \partial_\mu \varphi^\dagger). \tag{12}$$

Both currents are conserved using the equations of motion in eqn. (9) and its Hermitian conjugate.

The left and right currents are related by $\varphi \leftrightarrow \varphi^\dagger$ up to arbitrary normalization factors. Actually they share the same charge. It would then be tempting to introduce two other currents,

$$J_\mu^\pm = \frac{1}{2}\left(J_\mu^R \pm J_\mu^L\right), \tag{13}$$

which would correspond to the following two independent $U(1)$ transformations,

$$\begin{aligned} \varphi(x) &\to U_1(x) \star \varphi(x) \star U_1(x), \\ \varphi(x) &\to U_2(x) \star \varphi(x) \star U_2^{-1}(x), \end{aligned} \tag{14}$$

under which φ is charged and neutral respectively. The problem is that, when put together, it is ambiguous how to order the two transformations as mentioned above. Further, it is impossible to associate the first transformation with a real gauge boson which should transform as in eqn. (1). Thus, to gauge the symmetry consistently we should implement the transformation rule in eqn. (5) instead of the one in eqn. (14).

It is now standard to write down the Lagrangian for the doubly gauged φ^4 theory. The two gauge bosons are denoted as L_μ and R_μ with gauge couplings $g_{L,R}$. Together

with the φ transformation, we have

$$
\begin{aligned}
L_\mu &\rightarrow U_L(x)\star L_\mu \star U_L^{-1}(x) + \frac{i}{g_L}U_L(x)\star\partial_\mu U_L^{-1}(x),\\
R_\mu &\rightarrow U_R(x)\star R_\mu \star U_R^{-1}(x) + \frac{i}{g_R}U_R(x)\star\partial_\mu U_R^{-1}(x).
\end{aligned} \tag{15}
$$

The covariant derivative and field tensors are

$$
\begin{aligned}
D_\mu\varphi &= \partial_\mu\varphi - ig_L L_\mu\star\varphi + ig_R\varphi\star R_\mu,\\
L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu - ig_L[L_\mu, L_\nu]_\star,\\
R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu - ig_R[R_\mu, R_\nu]_\star,
\end{aligned} \tag{16}
$$

with $[f_1, f_2]_\star = f_1\star f_2 - f_2\star f_1$. Then,

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_\varphi + \mathcal{L}_L + \mathcal{L}_R,\\
\mathcal{L}_\varphi &= D_\mu\varphi\star(D^\mu\varphi)^\dagger - m^2\varphi\star\varphi^\dagger - \frac{\lambda}{2}\varphi\star\varphi^\dagger\star\varphi\star\varphi^\dagger,\\
\mathcal{L}_L &= -\frac{1}{4}L_{\mu\nu}\star L^{\mu\nu},\\
\mathcal{L}_R &= -\frac{1}{4}R_{\mu\nu}\star R^{\mu\nu}.
\end{aligned} \tag{17}
$$

The action $S = \int d^4x\mathcal{L}$ is gauge invariant but not the Lagrangian density itself,

$$
\mathcal{L}_\varphi \rightarrow U_L\star\mathcal{L}_\varphi\star U_L^{-1},\quad \mathcal{L}_L \rightarrow U_L\star\mathcal{L}_L\star U_L^{-1},\quad \mathcal{L}_R \rightarrow U_R\star\mathcal{L}_R\star U_R^{-1}. \tag{18}
$$

The apparent asymmetry for $\mathcal{L}_\varphi$ in the left and right transformations arises because we have arbitrarily ordered the φ field in such a way that φ comes first in each term of $\mathcal{L}_\varphi$. This is immaterial of course.

Now comes the question of how to interpret the above gauge theory. As we emphasized above, the left and right currents correspond to the same charge. In this sense the two gauge bosons L_μ and R_μ stand on the same footing. On the other hand, to a single *electric* charge we certainly expect to associate a single gauge field that can be called *the* electromagnetic field. To solve this dilemma, we note that what determines the electric charge of the φ field is its global transformation, for which there is no difference between ordinary and starred products. This implies that the physical fields can be correctly identified by going to the commutative limit of $\theta_{\mu\nu}\rightarrow 0$ in the Lagrangian density. In this limit, we have

$$
D_\mu\varphi \rightarrow \partial_\mu\varphi - ieA_\mu\varphi, \tag{19}
$$

where A_μ can be identified unambiguously with the electromagnetic field with the electric coupling $e = \sqrt{g_L^2 + g_R^2}$. The orthogonal B_μ field drops from the covariant derivative and

is thus not related with the electric charge. Both fields are linear combinations of the original gauge bosons,

$$\begin{pmatrix} A_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} L_\mu \\ R_\mu \end{pmatrix}, \tag{20}$$

with $c = g_L/e,\ s = g_R/e$. Note that in the above limit the gauge terms simplify into the free kinetic terms quadratic in $L_\mu,\ R_\mu$ and thus also in $A_\mu,\ B_\mu$, hence the B_μ becomes a decoupled, harmless free field. This also means in passing that we will need a genuine NC vertex to normalize the other coupling parameter c or s which does not show up in the commutative limit.

Once the physical fields are identified, we should reexpress in terms of them the Lagrangian on NC spacetime. The relevant pieces are,

$$\begin{array}{rcl} D_\mu\varphi & = & \partial_\mu\varphi - ie(c^2A_\mu\star\varphi + s^2\varphi\star A_\mu) + iecs[\varphi, B_\mu]_\star, \\ L_{\mu\nu} & = & c(\partial_\mu A_\nu - \partial_\nu A_\mu - iec^2[A_\mu, A_\nu]_\star) \\ & + & s(\partial_\mu B_\nu - \partial_\nu B_\mu - iecs[B_\mu, B_\nu]_\star) \\ & - & iec^2s([A_\mu, B_\nu]_\star + [B_\mu, A_\nu]_\star), \\ R_{\mu\nu} & = & c(\partial_\mu B_\nu - \partial_\nu B_\mu - iecs[B_\mu, B_\nu]_\star) \\ & - & s(\partial_\mu A_\nu - \partial_\nu A_\mu + ies^2[A_\mu, A_\nu]_\star) \\ & + & iecs^2([A_\mu, B_\nu]_\star + [B_\mu, A_\nu]_\star). \end{array} \tag{21}$$

We notice some features in the Lagrangian formed from the above pieces. With $c = 0,\ s = 1$ or $c = 1,\ s = 0$ we recover a theory that would have been derived by following the transformation rules for $\psi_\pm$, plus a vector field B_μ that is completely free. Except for these two special cases, the theory is very different. First, the photon-matter interaction terms in $D_\mu\varphi$ appear in both orders with generally different weights, which makes the interactions even richer than the usual ones. Second, when omitting the mixing terms in the gauge terms, we find that the B terms are in a canonical form as would be obtained by gauging a $U(1)$ directly. But there is no way for the A terms to become canonical. This is consistent with our discussions following eqn. (14). That is to say, the symmetry properties are simplest in terms of the L_μ, R_μ fields which the physical interpretation can be better expressed in terms of the A_μ, B_μ fields.

To complete the gauge theory, we should add the gauge fixing terms and the corresponding ghost terms. They can be obtained by generalizing directly the formalism of Kugo and Ojima on ordinary spacetime [8] which utilizes Hermitian ghost and anti-ghost

fields,

$$\begin{aligned}\mathcal{L}_{\text{g.f.}}+\mathcal{L}_{\text{ghost}} &= i\mathsf{s}\left(-\frac{1}{2}(\xi_L\bar{c}_L\star h_L+\xi_R\bar{c}_R\star h_R)+(\partial^\mu\bar{c}_L)\star L_\mu+(\partial^\mu\bar{c}_R)\star R_\mu\right)\\ &= -(\partial^\mu h_L\star L_\mu+\partial^\mu h_R\star R_\mu)+\frac{1}{2}\left(\xi_L h_L\star h_L+\xi_R h_R\star h_R\right)\\ &\quad -i(\partial^\mu\bar{c}_L)\star D_\mu c_L-i(\partial^\mu\bar{c}_R)\star D_\mu c_R,\end{aligned}\tag{22}$$

where $h_{L,R}$ are auxiliary fields, $\xi_{L,R}$ are gauge parameters, and $D_\mu c_L=\partial_\mu c_L+ig_L[c_L,L_\mu]_\star$, $D_\mu c_R=\partial_\mu c_R+ig_R[c_R,R_\mu]_\star$. The nilpotent BRS variations are ($j=L,R$),

$$\begin{aligned}\mathsf{s}\varphi &= ig_Lc_L\star\varphi-ig_R\varphi\star c_R,\\ \mathsf{s}\varphi^\dagger &= ig_Rc_R\star\varphi^\dagger-ig_L\varphi^\dagger\star c_L,\\ \mathsf{s}L_\mu &= D_\mu c_L,\\ \mathsf{s}R_\mu &= D_\mu c_R,\\ \mathsf{s}c_j &= ig_jc_j\star c_j,\\ \mathsf{s}\bar{c}_j &= ih_j,\\ \mathsf{s}h_j &= 0.\end{aligned}\tag{23}$$

The BRS currents and charge can also be constructed. We found that the BRS currents generally contain terms which are not in a closed form in terms of star product but are cumbersome series in the NC parameter. However, when time is not involved in the noncommutativity, all these terms do not contribute to the conserved BRS charge,

$$Q_{\text{BRS}} = \int d^3\mathbf{x}\sum_{j=L,R}\left(-\dot{h}_j\star c_j+h_j\star D_0c_j+g_j\dot{\bar{c}}_j\star c_j\star c_j\right)(x),\tag{24}$$

where the dot stands for the time derivative. Using the canonical equal-time (anti-) commutation relations we have verified that the above charge indeed generates the BRS variations shown in eqn. (23). The physical Hilbert space is then characterized by its annihilation by the BRS charge.

In summary, we have proposed that a matter field charged under a single $U(1)$ allows for a starred local transformation rule on NC space that acts from the left and right independently. This makes possible a double gauging of one global $U(1)$ symmetry on NC space. We suggested how the resulting theory should be interpreted in terms of physical degrees of freedom that are identified using the global property of charge: one gauge boson interacts with the charge while the other interacts only noncommutatively and thus decouples on ordinary space. We have also shown that the interactions in this theory have a richer structure than those obtained by a direct gauging of the charge $U(1)$ symmetry.

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